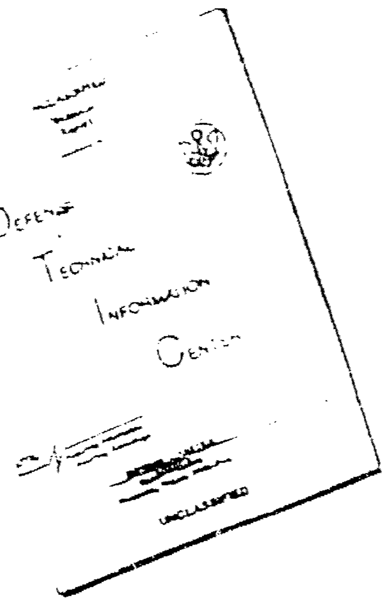

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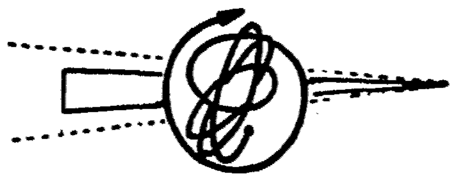
John D. Nicolaides



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TWO NON-LINEAR PROBLEMS IN THE FLIGHT*
DYNAMICS OF MODERN BALLISTIC MISSILES

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Department of the Navy

ABSTRACT

While the linear theory has been the backbone of ballistic missile design and has served with excellent success in its various forms over the years, recent spectacular missile flight failures have appeared unexpectedly and are unaccounted for by this linear theory. It is these flight failures which now require attention and consideration of the non-linearities in the fluid force and moment system which contribute to missile flight performance and dynamic stability.

Two types of non-linear flight instability have been isolated and identified as:

1. Non-Linear Magnus Instability, and
2. Catastrophic Yaw.

It is these two flight instabilities which will be discussed in the sections which follow. Approximate mathematical models will be suggested for the evaluation of missile dynamic stability. Both experimental and analytical applications of these models will be made and discussed.

*Originally contained in a paper prepared for Dr. Max M. Munk, Catholic University of America.

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INTRODUCTION

In recent years some conventional ballistic weapons or ordnance have exhibited flight performance failures not predicted by the Unified Linear Theory.[1-19] In all cases the initial conditions at launching were large due either to launcher motion as exemplified by subsonic rockets fired broadside from a moving ship and by bombs dropped from high speed aircraft, or to large launching tolerances as exemplified by excessive gun or mortar bore clearance. The effect of these initial launching conditions was to produce large mean values of the complex angle of attack. It was, therefore, suspected that the missile flight difficulties were due to non-linearities in the fluid force and moment system with complex angle of attack. The actual missile flight failures themselves fell into two distinct groups, one characterized by missiles having large rolling velocity and the other by missiles having small rolling velocity. These two groups will be treated separately in the sections which follow since quite different and relatively independent non-linearities are involved.

NON-LINEAR MAGNUS INSTABILITY

When certain subsonic fin-stabilized rockets having high spin were fired cross-wind from a moving ship, large and persistent wobbling motions occurred.[20] It was suspected, by an examination of the linear criterion for dynamic stability,[21-30] that a non-linear Magnus moment with complex angle of attack might be the culprit. Acting upon this conjecture a minor configurational change was introduced to reduce the Magnus moment.[31-33] Subsequent track firing results indicated that the flight instability was reduced to tolerable limits and, thus, this critical weapon was released to the fleet for full service use.

This experience had, however, uncovered a possible non-linear Magnus instability which required best study and understanding. Accordingly, an extensive wind tunnel program was undertaken by the Naval Ordnance Laboratory, a non-linear computation program was undertaken by the Naval Proving Ground, and an integrated flight testing and evaluation program was undertaken at the Naval Ordnance Test Station.

A simple empirical method for predicting this non-linear Magnus instability was suggested by the writer and was employed as a guide for actually eliminating a number of missile flight failures of this type. One of the purposes of this paper, therefore, is to set forth this method and to suggest a mathematical model for substantiating it.

Quasi Linear Theory

Flight instabilities in ballistic missiles are characterized by an almost pure circular pitching and yawing motion which grows to large values.

In the linear case this motion is easily traced to an instability in either the nutational or precessional components. It is the non-linear case which now requires attention. As a starting point it seems reasonable to assume "a priori" that the missile has an almost pure circular pitching and yawing motion. The dynamic stability of the missile may, then, be examined with reference to various sizes of this special motion. (Fig. 1) It should be emphasized that this happy state of affairs of almost constant magnitude of the complex angle of attack exists in rolling fin-stabilized missiles and shells but does not exist in the case of the pure pitching or pure yawing motion of non-rolling aircraft. It may be for this reason that this approach has not been considered previously in missile flight.

The first use of this assumption will be in specifying the aeroballistic force and moment system. All the forces due to angle of attack will be specified by a constant force plus a perturbation. (Fig. 2) Two forces and their moments are allowed to be non-linear. These forces are the normal force due to angle of attack and the Magnus force due to angle of attack and rolling velocity. All other aerodynamic forces and moments are assumed to be linear in their respective variables. Accordingly, the normal fluid force and the normal fluid moment acting on a missile* executing an almost pure circular pitching and yawing motion may be written as

$$\vec{Z} = \left(\hat{Z}_w W + \hat{Z}_{\dot{w}} \dot{w} \right) \frac{\vec{w}}{|\vec{w}|} + Z_{\vec{\xi}} \vec{\xi} + Z_{\dot{\vec{w}}} \dot{\vec{w}} + Z_{\dot{\vec{\xi}}} \dot{\vec{\xi}} \qquad (1)$$

$$\vec{M} = \left(\hat{M}_w W + \hat{M}_{\dot{w}} \dot{w} \right) \frac{\vec{w}}{|\vec{w}|} + M_{\vec{\xi}} \vec{\xi} + M_{\dot{\vec{w}}} \dot{\vec{w}} + M_{\dot{\vec{\xi}}} \dot{\vec{\xi}} \qquad (2)$$

*Rotational and mirror symmetry is assumed.

where

$$
\begin{aligned}
\hat{Z}_w &= Z_w + i p Z_{pv} & \hat{M}_w &= p M_{pv} - i M_w \\
\hat{Z}_{w} &= Z_{w} + i p Z_{pw} & \hat{M}_{w} &= p M_{pw} - i M_{w} \\
Z_{\bar{q}} &= p Z_{pq} + i Z_q & M_{\bar{q}} &= M_q - i p M_{pq} \\
Z_{\dot{w}} &= Z_{\dot{w}} + i p Z_{p\dot{w}} & M_{\dot{w}} &= p M_{p\dot{w}} - i M_{\dot{w}} \\
Z_{\dot{q}} &= p Z_{p\dot{q}} + i Z_{\dot{q}} & M_{\dot{q}} &= M_{\dot{q}} - i p M_{p\dot{q}}
\end{aligned} \tag{3}
$$

$$
|\vec{w}| = W + w \qquad W >> w \tag{4}
$$

The basic differential equations of motion written in terms of Aeroballistic Axes (Fig. 3) are given by*

$$
(\hat{Z}_w W + \hat{Z}_w w)\frac{\vec{w}}{|\vec{w}|} = m(\dot{\vec{w}} - i u \vec{q}) \tag{5}
$$

$$
(M_w W + \hat{M}_w w)\frac{\vec{w}}{|\vec{w}|} + M_{\bar{q}} \vec{q} + M_{\dot{w}} \dot{\vec{w}} = I \dot{\vec{q}} - i p I_x \vec{q} \tag{6}
$$

Solving Eq. (5) for $\vec{q}$, differentiating for $\dot{\vec{q}}$, and substituting into Eq. (6) yields a single differential equation in complex $\vec{w}$ which by substitution of Eq. (7), $\vec{w} = (W + w)e^{i\theta}$, and its derivatives may be reduced to a differential equation in real w as

$$
\ddot{w} + \vec{N}_1 \dot{w} + \vec{N}_2 w = \vec{N}_3 \tag{8}
$$

Eq. (8) may be written with real coefficients as

$$
\ddot{w} + n_1 \dot{w} + n_2 w = n_3 \tag{9}
$$

provided that the condition imposed by Eq. (10) is satisfied.

$$
\left[R\vec{N}_2 - \frac{R\vec{N}_3}{J\vec{N}_3} J\vec{N}_2 \right](J\vec{N}_1) = 0 \tag{10}
$$

---

*For convenience we shall let

$p = \text{const} \quad u = \text{const} \qquad Z_{\bar{q}} = Z_{\dot{w}} = Z_{\dot{q}} = M_{\dot{q}} = 0$

where

$$m_1 = \mathcal{R}\bar{N}_1 + \mathcal{J}\bar{N}_1 = \left(-\frac{Z_w}{m} - \frac{M_q + uM_{\dot w}}{I}\right) + \left(-p\frac{Z_{pw}}{m} + 2\dot\theta - p\frac{I_x}{I} + p\frac{M_{px}}{I} - \frac{upM_{p\dot w}}{I}\right)$$ (11)

$$m_2 = \mathcal{R}\bar{N}_2 + \mathcal{J}\bar{N}_2 = \left(\frac{pZ_{pw}}{m}\dot\theta - \dot\theta^2 - \frac{pI_x Z_{pw}}{I\,m} + \frac{pI_x}{I}\dot\theta + \frac{M_q Z_w}{I\,m} + \frac{p^2 M_{px} Z_{pw}}{I\,m} - \frac{pM_{px}}{I}\dot\theta - \frac{uM_w}{I} + \frac{upM_{p\dot w}}{I}\dot\theta\right)$$
$$+\left(\ddot\theta - \frac{Z_w}{m}\dot\theta + \frac{pI_x Z_w}{I\,m} - \frac{pZ_w M_{px}}{m\,I} + \frac{pZ_{pw}M_q}{m\,I} - \frac{M_q}{I}\dot\theta - \frac{upM_{pw}}{I} - \frac{uM_{\dot w}}{I}\dot\theta\right)$$ (12)

$$m_3 = \mathcal{R}\bar{N}_3 + \mathcal{J}\bar{N}_3 = \left(-\frac{pZ_{pw}}{m}\dot\theta + \frac{p^2 I_x Z_{pw}}{I\,m} + \frac{uM_w}{I} + \dot\theta^2 - \frac{pI_x\dot\theta}{I} - \frac{M_q Z_w}{I\,m} - \frac{p^2 M_{px} Z_{pw}}{I\,m} + \frac{pM_{px}\dot\theta}{I} - \frac{upM_{p\dot w}}{I}\dot\theta\right)W$$
$$+\left(\frac{Z_w}{m}\dot\theta - \frac{pI_x Z_w}{I\,m} + \frac{upM_{pw}}{I} + \frac{pZ_w M_{px}}{m\,I} - \frac{pZ_{pw}M_q}{m\,I} + \frac{M_q}{I}\dot\theta + \frac{uM_{\dot w}}{I}\dot\theta - \ddot\theta\right)W$$ (13)

The condition of Eq. (10) may be satisfied either if $\mathcal{J}N_1 = 0$ or if the quantity in the bracket is equal to zero (i.e. if $\mathcal{R}N_2 = 0$ and if $\mathcal{R}N_3 = 0$). Considering the latter situation and writing the expression for $\mathcal{R}N_2$ as,

$$\mathcal{R}N_2 = -\left[\dot\theta^2 + p\left(\frac{I_x}{I} - \frac{M_{px}}{I} + \frac{Z_{pw}}{m} + \frac{uM_{p\dot w}}{I}\right)\dot\theta + \left(\frac{uM_w}{I} + \frac{p^2 I_x Z_{pw}}{I\,m} - \frac{M_q Z_w}{I\,m} - \frac{p^2 M_{px} Z_{pw}}{I\,m}\right)\right]$$
$$= -\left(\dot\theta - \Omega_1\right)\left(\dot\theta - \Omega_2\right)$$ (14)

where (15)

$$\Omega_{1,2} = \frac{pI_x}{2I}(1 \pm \tau) - \frac{pM_{px}}{2I}(1 \pm \tau) + \frac{pZ_{pw}}{2m}(1 \pm \tau) + \frac{upM_{p\dot w}}{2I}(1 \pm \tau)$$ (16)

$$\approx \frac{pI_x}{2I}(1 \pm \tau)$$ (17)

$$\tau = \sqrt{1 - \frac{1}{s}}, \qquad s = \frac{I\left(\frac{pI_x}{I}\right)^2}{4uM_w}$$ (18)

it may be noted that $RN_2 = 0$, if $\dot{\phi} = r_1$, or if $\dot{\phi} = r_2$.

Writing the expression for RN_3 as,

$$RN_3 = \left[\dot{\theta}^2 - p\left(\frac{I_x}{I} - \frac{M_{pw}}{I} + \frac{Z_w V}{m} + u\frac{M_{pw}}{I}\right)\dot{\theta} + \left(\frac{u M_w}{I} + \frac{2p^2 I_x Z_{pv}}{2Im} - \frac{M_q Z_w}{I\,m} - p^2\frac{M_{pw} Z_{pw}}{I\,m}\right)\right] \tag{19}$$

and assuming that

$$Z_{pv} = Z_{pw} \qquad M_w = M_{w} \tag{20}$$

yields

$$RN_3 \simeq (\dot{\theta} - r_1)(\dot{\theta} - r_2) \tag{21}$$

Therefore, it may be noted from Eqs. (15) and (21) that the condition imposed by Eq. (10) may be satisfied if

$$\dot{\theta} = r_1 \qquad \text{or} \qquad \dot{\theta} = r_2 \tag{22}$$

provided that the Magnus force and the static moment are linear, Eq. (20).

We may now return to the basic differential equation of motion, Eq. (9). Assuming that $\ddot{\phi} = 0$, the general solution is given by

$$\underline{w} = \underline{K}_1 e^{\underline{\phi}_1 t} + \underline{K}_2 e^{\underline{\phi}_2 t} + \underline{K}_3 \tag{23}$$

where

$$\underline{\Phi}_{1,2} = \left(-\frac{m_1}{2}\right) \pm \left\{\frac{m_1^2}{4} - m_2\right\}^{1/2} \tag{24}$$

$$\underline{K}_{1,2} = \frac{\dot{\underline{w}}_0 - \underline{\phi}_{2,1}\underline{w}_0 + \underline{\phi}_{2,1}\underline{K}_3}{\underline{\phi}_{1,2} - \underline{\phi}_{2,1}} \tag{25}$$

$$\underline{K}_3 = \frac{m_3}{m_2} \tag{26}$$

Before considering in detail the solution given by Eq. (23), it is desirable to obtain expressions for $\underline{\phi}_{1,2}$ and $\underline{\lambda}_{1,2,3}$ in terms of the stability derivatives. Accordingly, $\underline{\phi}_{1,2}$ is given by,

$$\underline{\Phi}_{1,2} = \left(\frac{Z_w}{2m} + \frac{(M_q + uM_{\dot{w}})}{2I} + p\frac{Z_{pv}}{2m} - \dot{\phi} + \frac{pI_x}{2I} - p\frac{M_{pu}}{2I} + u\frac{M_{p\dot{w}}}{2I} \right)$$

$$\pm \left\{ (\dot{\phi} - R_1)(\dot{\phi} - R_2) + \left(\dot{\phi} - \frac{pI_x}{2I}\right)^2 + p\left(\frac{M_{pu}}{I} - \frac{Z_{pv}}{m} - u\frac{M_{p\dot{w}}}{I} \right)\dot{\phi} \right.$$

$$\left. + \left(\frac{pI_x}{2I}(M_q + uM_{\dot{w}}) - \frac{pI_x Z_w}{2mI} - \frac{p^2 I_x M_{pu}}{2I^2} + \frac{p^2 I_x u M_{p\dot{w}}}{2I^2} + \frac{p^2 I_x Z_{pv}}{2mI} + up\frac{M_{pv}}{I} \right) \right\}^{1/2} \qquad (27)$$

This expression may be further simplified if the terms under the radical are divided into two groups, one containing terms which are large and the other containing the small terms. Introducing the approximation

$$\sqrt{L^2 + S^2} \approx L + \frac{S^2}{2L} \qquad (28)$$

Eq. (27) becomes,

$$\underline{\Phi}_{1,2} = \frac{Z_w}{2m}(1 \mp \tau) + \frac{M_q + uM_{\dot{w}}}{2I}(1 \pm \tau) \pm \frac{uM_{pv}}{I_x}\tau$$

$$+ \left(\frac{pI_x}{2I} - \dot{\phi}\right) + (1 \mp 1)\left(-\frac{pI_x}{2I\tau} + \frac{pZ_{pv}}{2m} - p\frac{M_{pu}}{2I} + \frac{up M_{p\dot{w}}}{2I} \right) \qquad (29)$$

$$\underline{\Phi}_{1,2} = \lambda_{1,2} + i\,\underline{\omega}_{1,2} \qquad (30)$$

where the condition imposed by Eq. (2.) has been satisfied. While $\underline{\phi}_{1,2}$ was originally allowed to be complex, it may be noted from Eq. (29) that it

is real. Two sets of values for $\lambda_{1,2}$ may be obtained depending upon which condition in Eq. ([illegible]) is satisfied, as

if $\dot{\phi} = r_1$, then

$$\lambda_1 = \frac{Z_w}{2m}(1-\tau) + \frac{M_q + uM_{\dot{w}}}{2I}(1+\tau) + \frac{uM_{pw}}{I_p}\tau \qquad (31)$$

$$\lambda_2 = \frac{Z_w}{2m}(1+\tau) + \frac{M_q + uM_{\dot{w}}}{2I}(1-\tau) - \frac{uM_{pw}}{I_p}\tau - \frac{pZ}{I}\tau + \frac{pZ_{pw}}{m} - \frac{M_{p\alpha}}{I} + \frac{upM_{pw}}{I} \qquad (32)$$

or, if $\dot{\phi} = r_2$, then

$$\lambda_1 = \frac{Z_w}{2m}(1-\tau) + \frac{M_q + uM_{\dot{w}}}{2I}(1+\tau) + \frac{uM_{pw}}{I_p}\tau + \frac{pZ}{I}\tau - \frac{pM_{p\alpha}}{I} + \frac{pZ_{pw}}{m} + \frac{upM_{pw}}{I} \qquad (33)$$

$$\lambda_2 = \frac{Z_w}{2m}(1+\tau) + \frac{M_q + uM_{\dot{w}}}{2I}(1-\tau) - \frac{uM_{pw}}{I_p}\tau \qquad (34)$$

An expression for K_3 in terms of the stability derivatives, when Eq. ([illegible]) is satisfied, is given by,

$$K_3 = -\left[\frac{\frac{Z_w}{2m}(1\mp\tau) + \frac{(M_q + uM_{\dot{w}})}{2I}(1\pm\tau) \pm \frac{uM_{pN}}{I_p}\tau}{\frac{Z_w}{2m}(1\mp\tau) + \frac{(M_q + uM_{\dot{w}})}{2I}(1\pm\tau) \pm \frac{uM_{pN}}{I_p}\tau}\right]_{\dot{\theta}=R_1,\ \dot{\phi}=R_2} W = -\left[\frac{\lambda^*_{1,2}}{\lambda_{1,2}}\right]_{\dot{\theta}=R_1,\ \dot{\phi}=R_2} W \qquad (35)$$

It should be noted that the denominator of the bracket quantity, $\lambda_{1,2}$, is identically the familiar linear theory expression for the nutation and precession damping rates where the tangent slopes are used. The [illegible]umerator on the other hand while identical in form uses the secant slopes for the non-linear quantities. (Fig. 2)

At the outset, in specifying the fluid forces, Eqs. (1) and (2), it was assumed that the missile had almost pure circular pitching and yawing motion. The selection of initial conditions in determining K_1 and K_2 should, therefore, be such as to satisfy at least initially this requirement. Thus, we assume that,

$$\underline{\omega} = \underline{\omega}_0 \qquad \dot{\underline{\omega}} = \dot{\underline{\omega}}_0 = 0 \tag{36}$$

The expression for $K_{1,2}$, Eq. (25), reduces to

$$K_{1,2} = \frac{-\lambda_{2,1}\left(\omega_0 - K_3\right)}{\lambda_{1,2} - \lambda_{2,1}} \tag{37}$$

In considering the stability of a missile, we are interested in the conditions under which the missile has only marginal dynamic stability. That is, we wish to find a bound below or above which the missile is either dynamically stable or unstable. Thus we are interested in studying Eq. (23) for conditions where λ_1 or λ_2 is approaching zero. It may be noted from Eqs. (31-34) that, usually, when one λ is small the other is large. Accordingly Eq. (37) may be written as,

$$\left.\begin{aligned} K_1 &= (\omega_0 - K_3) \\ K_2 &= -\frac{\lambda_1}{\lambda_2} K_1 \end{aligned}\right\} \quad \lambda_1 \to 0 \tag{38}$$

$$\left.\begin{aligned} K_1 &= -\frac{\lambda_2}{\lambda_1} K_2 \\ K_2 &= (\omega_0 - K_3) \end{aligned}\right\} \quad \lambda_2 \to 0 \tag{39}$$

Substituting these expressions and Eq. (4) back into Eq. (23) yields,

$$\underline{\omega} = (\omega_0 - K_3)\, e^{\lambda t} + (W + K_3) \tag{40}$$

where $\lambda = \lambda_1$ when $\lambda_1 \rightarrow 0$

or $\lambda = \lambda_2$ when $\lambda_2 \rightarrow 0$

Since r_1 is associated with λ_1, and r_2 with λ_2, we may simply write our solution for the motion as,

$$|\vec{w}| = \left(\underline{w}_o + \frac{\lambda^*}{\lambda} W \right) e^{\lambda t} + W_T \qquad \begin{cases} \text{for } \dot{\phi} = \dot{\phi}_1, & \lambda = \lambda_1 \rightarrow 0 \\ \text{or } \dot{\phi} = \dot{\phi}_2, & \lambda = \lambda_2 \rightarrow 0 \end{cases} \tag{41}$$

where

$$W_T = \left(1 - \frac{\lambda^*}{\lambda} \right) W \tag{42}$$

This is, therefore, the basic equation for the almost circular motion of a missile acted upon by a non-linear Magnus moment. In the following section this equation will be examined for evaluating the dynamic stability of the missile.

Discussion of the Quasi-Linear Theory for Magnus Instability

Our purpose here is to discuss Eq. (41) with a view towards determining both the conditions for missile dynamic stability and a measure of the amount of dynamic stability which a missile has for given flight conditions.

The condition for dynamic stability in the <u>linear</u> case[1] is given by,

$$0 > \lambda_{1,2} = \left[\frac{Z_{w}}{2m}(1 \mp \tau) + \frac{(M_q + uM_{\dot{w}})}{2I}(1 \pm \tau) \pm \frac{u\,M_{pw}}{I_x}\tau\right]_{W=0} \qquad (43)$$

Thus, as a starting point for studying Eq. (41), it seems reasonable to begin by examining the case of negative $\underline{\lambda}$. First, it may be noted that when $W=0$, then $|\vec{w}| \to 0$ and also when $\underline{\lambda} = \lambda^*$, $|\vec{w}| \to 0$. Both of these reduced cases are essentially the familiar linear one when only one arm exists.

Of basic interest in this non-linear solution, Eq. (41), is the fact that a trim-like condition exists which is represented by W_T. In considering missile stability, it is important to determine the size of this pseudo trim as compared to the basic motion, W, about which the perturbation, w, takes place. Thus, values for W_T/W are given by Eq. (42) and plotted in Fig. 4.

In the familiar linear case when $\lambda < 0$ dynamic stability was indicated. However, in this non-linear solution it may be noted that if $W_T > W$ and $\underline{\lambda} < 0$, then a divergence from the basic motion results thereby indicating dynamic instability. Of course, if $W_T < W$ and $\underline{\lambda} < 0$ dynamic stability results

The use of the terms "dynamic stability", etc. in this non-linear case should be defined. A basic motion represented by W is perturbed by the addition of $\bar{w}$, Fig. (3). If the resulting motion, $|\bar{w}|$, tends to reduce in size approaching W or a value less than W, then the motion is termed "dynamically stable" for that value of W.

The various possibilities for stability and instability in this non-linear case, Eq. (41), may be perceived by considering the two basic cases which are represented by $W_T < W$, Fig. 4, and $W_T > W$, Fig. 4. The former is obtained when λ^*/λ is positive and the latter when λ^*/λ is negative. (See Eq. (42) and Fig. 4.)

Two possibilities for positive λ^*/λ exist. When λ is negative and λ^* is negative, dynamic instability results. In Fig. 5 these two possibilities are represented. In the first possibility ($\lambda < 0$ and $\lambda^* < 0$), the transient part, $(\quad)e^{\lambda t}$, shrinks, and thus the total motion tends towards $W_T < W$, thus indicating dynamic stability. In the second possibility ($\lambda > 0$ and $\lambda^* > 0$) the transient part grows, and thus the total motion tends to increase away from W_T and W, indicating dynamic instability.

Two possibilities for negative λ^*/λ also exist. When λ is negative and λ^* is positive, the missile is dynamically unstable. However, when λ is positive and λ^* is negative, dynamic stability results. This latter possibility is of particular interest since it certainly would not be anticipated from any linear analysis. These two possibilities are represented in Fig. 6. In the first ($\lambda < 0$ and $\lambda^* > 0$), the transient part shrinks in size and thus the total motion grows away from W and

towards $W_T > W$, indicating instability. In the second ($\underline{\lambda} > 0$ and $\underline{\lambda}^* < 0$), the transient part grows in size and thus the total motion shrinks toward W, indicating stability.

When $\underline{\lambda}^* = 0$ and $\underline{\lambda}$ is negative, the pitching and yawing motion will be stable in the original pure circular mode neither damping nor expanding since $W_T = W$. The stable modes of circular motion (i.e. "Limit Cycles") have been previously investigated by other writers.[31, 35-38]

When $\lambda^* = 0$ and $\underline{\lambda}$ is positive the motion will grow and thus be unstable.

While negative values for W_T/W have not been specifically discussed or illustrated, all general statements still apply. For completeness, one typical case is represented in Fig. 7 & 8 and is now discussed. When $\underline{\lambda} < 0$ and $\underline{\lambda}^* < 0$ and $|\lambda^*| > |\lambda|$ then $W_T < 0$ and the transient motion reduces in size; the total motion tends toward W and $|W_T| < W$ and is, thus, dynamically stable.

In summary it has been shown that the missile is dynamically stable when perturbed from the basic motion for two cases when

$$(1) \quad \underline{\lambda} < 0 \quad \text{and} \quad \underline{\lambda}^* < 0 \tag{44}$$

$$(2) \quad \underline{\lambda} > 0 \quad \text{and} \quad \underline{\lambda}^* < 0 \tag{45}$$

, and the missile is dynamically unstable in three cases when

$$(1) \quad \underline{\lambda} < 0 \quad \text{and} \quad \underline{\lambda}^* > 0 \tag{46}$$

$$(2) \quad \underline{\lambda} > 0 \quad \text{and} \quad \underline{\lambda}^* > 0 \tag{47}$$

$$(3) \quad \underline{\lambda} < 0 \quad \text{and} \quad \underline{\lambda}^* = 0 \tag{48}$$

A Limit Cycle exists when $\underline{\lambda}^* = 0$ and $\underline{\lambda} < 0$.

Clearly the sign of $\underline{\lambda}$ is not a measure of missile dynamic stability in this non-linear case as it was in the linear case. In reviewing Eqs. (44) - (49) it is found that the sign of $\underline{\lambda}^{*}$ is, however, a criterion for dynamic stability. Thus we find, with some amazement, that the requirement for Dynamic Stability in this non-linear case is simply that

$$\underline{\lambda}^{*} < 0 \tag{50}$$

where

$$\underline{\lambda}^{*} = \frac{Z_w}{2m}(1 \mp \tau) + \frac{(M_q + u M_{\dot{w}})}{2I}(1 \pm \tau) \pm \frac{u M_{pV}}{I_x}\tau \tag{51}$$

This condition suggests that a plot of $\underline{\lambda}^{*}$ vs. $w^2 + v^2 = W$ will immediately reveal an evaluation of the dynamic stability of the missile. This is the empirical method which was originally suggested and which the preceding mathematical model seems to substantiate. By evaluating $\underline{\lambda}^{*}$ for each value of W using the actual non-linear aerodynamic coefficients, the plot is made. It should be recalled that two expressions for $\underline{\lambda}^{*}$ exist depending on whether $\dot{\phi} = r_1$ or $\dot{\phi} = r_2$. Thus, two plots should be made. In keeping with classical usage the faster circular motion will be called "nutation" and the slower, "precession". Accordingly, regions of missile non-linear dynamic stability may be revealed in such plots as given by Fig. 9.

Not only may the dynamic stability be determined but, also, the <u>amount</u> of dynamic stability may be evaluated by using the non-linear theory. In the stable case, by substituting $|\vec{w}| = \frac{|\vec{w}_0|}{2} = \frac{\underline{w}_0 + W}{2}$ into Eq. (41) yields an expression for Motion Half-Life as a function of W as

$$\mathcal{D}^{(h)}_{N,P} = \frac{V}{d\,\lambda_{N,P}} \ln\left[\frac{\underline{w}_0 + W\left(2\frac{\lambda^*}{\lambda} - 1\right)}{2\left(\underline{w}_0 + \frac{\lambda^*}{\lambda}W\right)}\right]_{N,P} \tag{52}$$

$$\approx \frac{V}{d\,\lambda_{N,P}} \ln\left[1 - \frac{\lambda_{N,P}}{2\lambda^*_{N,P}}\right]_{\underline{w}_0 \ll W} \tag{53}$$

In the unstable case, an expression for Motion Double-Life may be obtained by substituting $|\vec{w}| = 2\,|\vec{w}_0|$ into Eq. (41) as

$$\mathcal{D}^{(2)}_{N,P} = \frac{V}{d\,\lambda_{N,P}} \ln\left[\frac{2\underline{w}_0 + \left(1 + \frac{\lambda^*}{\lambda}\right)W}{\underline{w}_0 + \frac{\lambda^*}{\lambda}W}\right]_{N,P} \tag{54}$$

$$\approx \frac{V}{d\,\lambda_{N,P}} \ln\left[1 + \frac{\lambda_{N,P}}{\lambda^*_{N,P}}\right]_{\underline{w}_0 \ll W} \tag{55}$$

> As a result of the preceding analysis it has been found that:
>
> (1) the condition for dynamic stability is $\lambda^*(W) < 0$
>
> (2) the amount of dynamic stability is given by the expressions for Motion Half-Life and Double-Life.

Applications of the Quasi Linear Theory

Applications of the Quasi Linear Theory for Magnus Instability have been made to certain missiles which, in flight, exhibited a possible Magnus instability. The approach employed was, first, to make plots of $\lambda^*_{M,P}(W)$ for the original design, for modified designs, and for modified launching conditions and, then, select those conditions for which positive values for $\lambda^*_{M,P}$ were entirely avoided. Subsequent designs based on this procedure were all observed to be dynamically stable in flight. While the spectacular success of these limited experimental applications of the theory indicate promise, the full extent of its usefulness and of its limitations is essentially unknown. Since we are dealing with a highly non-linear problem, great caution and care should always be exercised.

In addition to the experimental applications, comparisons of the stability predictions of this theory were made with results from numerical integrations of the exact non-linear differential equations of motion on the Naval Ordnance Research Computer at the Naval Proving Ground, Dahlgren, Virginia. These comparisons of the predictions of the Quasi Linear Theory with the NORC results are given by Dr. C. J. Cohen and E. C. Hubbard in NPG Report No. 1421[39] to which the reader is referred.

Finally it should be emphasized that the Quasi-Linear Theory is not intended to predict the general detailed free flight motion of missiles, particularly when both nutation and precession arms are present. Rather, the theory suggests a method for quickly and easily evaluating missile dynamic stability when acted upon by a non-linear Magnus moment. For initial design

evaluations and as a guide for "quick fixes", it is strongly recommended; however, it is not intended to replace final numerical integrations of the exact differential equations of motion on a high speed computer. Experience, thus far, has indicated that the best results and a full physical understanding are obtained by using both.

CATASTROPHIC YAW THEORY

Introduction

A unique flight instability not accounted for by the Unified Linear Theory or by the Quasi Linear Theory was occasionally observed in the flights of a certain fin-stabilized missile.[40,41] This particular dynamic instability, when it occurred, was first characterized by the failure of the missile to pick up its full steady state rolling velocity and, then, by a catastrophic growth of the pitching and yawing motion. Initially the rolling velocity increases due to the fin cant. However, when it reaches a value equal to the nutation frequency it holds constant and "locks-in" at that particular value instead of seeking the much larger steady state value. (Fig.10) Initially the size of the pitching and yawing motion reduces. However, when the rolling motion locks-in, it begins to grow and may very soon reach extreme values such that the missile in flight looks more like a propeller than an arrow. (Fig. 11) This unique flat spin has been termed "Catastrophic Yaw".[42]

In attempting to find a physical explanation and possibly a theory for the prediction and future avoidance of the Catastrophic Yaw, the rolling motion will be considered first.

Rolling Motion

The rolling motion of missiles has been classically considered the simplest mode. Its knowledge and control, however, is essential in the guided missile in order to achieve guidance and in the ballistic missile

in order to achieve flight stability and accuracy. For missiles flying at small angles of attack, the linear theory has performed admirably.[43,44] In order to critically investigate the possible existence and nature of abnormal rolling motions, praticularly of the type observed in catastrophic flight, a simple fin-stabilized model (Basic Finner) was sting-mounted free to roll in the National Bureau of Standards wind tunnel.[45] The variation of the steady state rolling velocity with angle of attack for angles from 0° to 90° was measured. Typical data are given in Fig. 12 where it is noted that there are large and almost discontinuous changes in the steady state rolling velocity with angle of attack. Also, it is noted that the missile may roll in either direction and, if the missile is stopped, it may remain stopped.

For convenience the two major abnormal rolling motions observed during these exploratory tests are labeled "roll lock-in" and "roll speed-up".[46] Since, over a certain range of angles of attack, the missile can perform either motion, it was suspected that the two phenomena are basically independent and may be treated separately. For our present purposes, it is the failure of the missile to roll (roll lock-in) which required analysis and explanation.

Two key factors characterizing roll lock-in are gleaned from the experimental data. First, there is a definite dependence on both the angle of attack and the fin cant angle. Below a critical value of angle of attack, the model will roll; above that value, the model may fail to roll. In the range of critical values for the angle of attack, there is a critical value of the fin cant angle below which the model will fail to roll and above which the model will roll. Secondly, when the missile fails

to roll, the roll orientation of the model with respect to the plane of the angle of attack, γ, varies with the angle of attack. At the critical angle of attack, the roll orientation of the model is in the neighborhood of 67 1/2°. As the angle of attack is increased the roll orientation tends toward 45° as a limit.

These two characteristics of roll lock-in suggest the possible existence of a roll moment which depends on angle of attack and missile roll orientation. A review of the literature reveals that such a non-linear roll moment is known to exist and has been measured quite early on airships, torpedoes, and aircraft.[47,48] For a cruciform finned missile, this "Induced Roll Moment" takes the general form

$$L(\gamma\alpha) = C_{l_{\gamma\alpha}}\, \alpha\, (\sin 4\gamma)\, \tfrac{1}{2}\rho V^2 S d \tag{56}$$

$$C_{l_{\gamma\alpha}} \sim \gamma, \alpha$$

and is plotted in Fig. 13.

We are interested in determining if the addition of this non-linear roll moment to the linear roll equation can account for the phenomena of roll lock-in. Accordingly, the equation for the rolling motion may be rewritten as

$$L(\delta_n) + L(p) + L(\gamma,\alpha) = I_x \dot{p} \tag{57}$$

where

$L(\delta_n)$ = Roll Moment due to cant $= C_{l_{\delta_n}}(\alpha)\, \delta_n\, \tfrac{1}{2}\rho V^2 S d$ (58)

$L(p)$ = Roll Damping Moment $= C_{l_p}(\alpha, p)\left[\frac{pd}{2V}\right]\tfrac{1}{2}\rho V^2 S d$ (59)

$L(\gamma,\alpha)$ = Induced Roll Moment $= C_{l_{\gamma\alpha}}(\alpha,\gamma)\, \alpha\, [\sin 4\gamma]\, \tfrac{1}{2}\rho V^2 S d$ (60)

In the wind tunnel case of roll lock-in $p = 0$, $\dot{p} = 0$, and α = const. Thus Eq. (57) reduces to

$$L(\delta_a) + L(\gamma, \alpha) = 0 \tag{61}$$

In any particular wind tunnel test of roll lock-in,

$$\delta_a = \text{const} \qquad \alpha = \text{const}$$

Thus to satisfy Eq. (61) it is only necessary to find a value of γ for which

$$\gamma = \frac{1}{4} \sin^{-1}\left[-\frac{C_{l_{\delta_a}} \delta_a}{C_{l_{\gamma\alpha}} \alpha}\right] \tag{62}$$

The three possible non-trivial cases are illustrated in Fig. (14).

In Case 1 the roll moment due to cant is greater than the induced roll moment for all values of the roll orientation. Thus, there is no value of α satisfying Eq. (62) and therefore the model cannot lock-in and will roll.

In Case 2 the two roll moments are equal and opposite at $\gamma = 67\ 1/2^\circ$, and thus Eq. (62) is satisfied and the model can lock-in. This is seen to be the "critical angle of attack" for that particular roll moment due to cant. If the angle of attack is slightly decreased it is noted that the model cannot lock-in since we have Case 1 again. If the angle of attack is increased then we have another example of Case 3.

In Case 3 there are two values of γ for which Eq. (62) may be satisfied. For the value $67\ 1/2^\circ < \gamma \leq 90^\circ$ static roll instability exists since a slight increase in γ yields a roll moment due to cant greater than the induced roll moment which tends to cause a greater increase in γ and thus further static unbalance. If γ is slightly

reduced a similar unstable situation exists. However, values of $45^\circ \leq \gamma \leq 67\ 1/2^\circ$ are seen to be statically stable and thus roll lock-in is expected in this range.

The critical angle of attack for roll lock-in and the change in roll trim angle are thus given by Eq. (61) by simply inserting the physical parameters involved,

$$\alpha_{crit} = -\frac{C_{l_{\delta_a}}\,\delta_a}{C_{l_{\gamma\alpha}}} \tag{63}$$

$$\gamma_{TRIM} = \frac{1}{4}\sin^{-1}\left[-\frac{C_{l_{\delta_a}}\,\delta_a}{C_{l_{\gamma\alpha}}\,\alpha}\right] \qquad 45^\circ \leq \gamma_{TRIM} \leq 67\tfrac{1}{2}^\circ \tag{64}$$

where $C_{l_{\delta_a}}$ and $C_{l_{\gamma\alpha}}$ may both be non-linear in α and γ.

It was also observed in the wind tunnel tests that the model when excited had small oscillations in roll about the roll trim angle. The frequency of these oscillations may be predicted from Eq. (57) by assuming a linear variation in $L_r(\gamma,\alpha)$ for the small γ range involved. Accordingly Eq. (57) may be written as

$$\ddot{\gamma} + \tilde{N}_1\,\dot{\gamma} + \tilde{N}_2\,\gamma = \tilde{N}_3 \tag{65}$$

where

$$\tilde{N}_1 = -\frac{1}{I_x}\left[C_{l_p}\left(\frac{d}{2V}\right)\tfrac{1}{2}\rho V^2 S d\right] \tag{66}$$

$$\tilde{N}_2 = -\frac{1}{I_x}\left[\bar{C}_{l_{\gamma\alpha}}\,\alpha\,\tfrac{1}{2}\rho V^2 S d\right] \qquad \bar{C}_{l_{\gamma\alpha}} = \text{local slope at } \alpha = \text{const} \tag{67}$$

$$\tilde{N}_3 = +\frac{1}{I_x}\left[C_{l_{\delta_a}}\,\delta_a\,\tfrac{1}{2}\rho V^2 S d\right] \tag{68}$$

The general solution of Eq. (65) is given by

$$\gamma = \bar{\bar{K}}_1 e^{\bar{\bar{\varphi}}_1 t} + \bar{\bar{K}}_2 e^{\bar{\bar{\varphi}}_2 t} + \bar{\bar{K}}_3 \tag{69}$$

$$\bar{\bar{\varphi}}_{1,2} = -\frac{\bar{\bar{N}}_1}{2} \pm \frac{1}{2}\sqrt{\bar{\bar{N}}_1^{\,2} - 4\bar{\bar{N}}_2} = \bar{\bar{\lambda}}_{1,2} + i\,\bar{\bar{\omega}}_{1,2} \tag{70}$$

$$\bar{\bar{K}}_{1,2} = \frac{\dot{\gamma}_0 - \bar{\bar{\varphi}}_{2,1}\gamma_0 + \bar{\bar{K}}_3(\bar{\bar{\varphi}}_{2,1})}{\bar{\bar{\varphi}}_{1,2} - \bar{\bar{\varphi}}_{2,1}} \qquad \bar{\bar{K}}_3 = \frac{\bar{\bar{N}}_3}{\bar{\bar{N}}_2} \tag{71, 72}$$

The frequency of the roll oscillations is given by

$$\bar{\bar{\omega}}_{1,2} = \pm\left(-\frac{1}{I_x}\bar{C}_{l_{\gamma\alpha}}\,\alpha\,\tfrac{1}{2}\rho V^2 S d\right)^{1/2} \tag{73}$$

and the damping by

$$\bar{\bar{\lambda}}_{1,2} = \frac{1}{2 I_x}\left(C_{l_p}\left(\frac{d}{2V}\right)\tfrac{1}{2}\rho V^2 S d\right) \tag{74}$$

Sometimes, particularly at very large angles of attack, the model was observed to oscillate in a divergent fashion and begin to roll by itself. This type of roll performance is also accounted for by Eq. (69) if the roll damping moment is positive or undamping. The above analysis of the experimentally observed divergent roll oscillations suggests that the roll damping moment may be non-linear in the angle of attack and the rolling velocity of the missile. A non-linearity of this type might well account for the roll speed-up phenomena mentioned at the outset but not specifically studied here.[49]

Thus, the introduction of the non-linear induced roll moment into the classical linear roll theory which contains the roll moment due to cant and the roll moment due to rolling velocity appears to yield an explanation of and a prediction method for the roll lock-in observed in the wind tunnel tests.

Now that the wind tunnel roll lock-in phenomena appears to be accounted for, the free flight missile roll case might be profitably considered. This missile case presents two additional complications. First, the missile is rolling and second, the missile's angle of attack may be changing. It is recalled that in the missile case the rolling velocity failed to increase to its design value and it stopped increasing at a value of rolling velocity equal to the frequency of the missile's pitching and yawing motion. Now when a missile is rolling at the same rate that it is wobbling and if the wobbling is in the same sense as the rolling then we have the special case where the missile is not changing its roll orientation with respect to the plane of the complex angle of attack or, said otherwise, γ is a constant. This type of pitching and yawing motion is called "Lunar Motion"[50,51] because of its analogy to the motion of the moon.

If pure lunar pitching and yawing motion exists, where the angle of attack is a constant, then the analysis used in the wind tunnel case may readily be extended to the missile case by simply adding the roll damping moment.

Eq. (57) may be written as

$$L(\delta_A) + L(p) + L(\gamma, \alpha) = 0 \tag{75}$$

where δ_A, p, α, and γ are constants, the critical angle of attack is given by

$$\alpha_{CRIT} = -\left[\frac{C_{l_{\delta_A}}\delta_A - C_{l_p}\left(\frac{pd}{2V}\right)}{C_{l_{\gamma\alpha}} \sin 4\gamma}\right] \tag{76}$$

and the roll trim angle is given by

$$\gamma_{TRIM} = \frac{1}{4} \sin^{-1} \left[- \frac{1}{\alpha C_{l_{\gamma\alpha}}} \left(C_{l_{\delta_a}} \delta_a - C_{l_p} \left(\frac{pd}{2V} \right) \right) \right] \qquad (77)$$

In Fig. 14 the solution to Eq. (77) is represented.

The phenomenon of roll lock-in as observed in the special wind tunnel tests and in full scale free flight of a particular missile appears to be traceable to the actions of the non-linear induced roll moment. When this moment is introduced into the classical theory for the pure rolling motion, a method for predicting the critical angle of attack and the roll trim angle is evolved.

Pitching and Yawing Mo[illegible]

It now remains to answer the second question, "why the catastrophic growth of the pitching and yawing motion?" It was established in the previous section and borne out by the flight performance data that when roll lock-in occurred, the pitching and yawing motion is of the "Lunar Type". Specifically then, we are concerned about the catastrophic growth of lunar motion. For a statically stable missile, the Unified Linear Theory indicates that lunar motion may occur in two distinct cases. One, when pure nutation (i.e. K_1) exists and the nutation rate is equal to the roll rate; the other, when pure trim (i.e. K_3) exists. In the latter case lunar motion exists for all values of the missile rolling velocity.

Of course, a combination of both of the cases yielding lunar motion can exist when the rolling velocity is equal to the nutation rate. Although the combination may be more realistic, we shall first consider the two possibilities separately.

The dynamic stability of pure nutational motion in the linear and in the non-linear case was treated in the previous sections. Applying these methods of prediction to the missile in question completely fails to account for the observed catastrophic growth. The size of the trim is well known to be a function of the rolling velocity of the missile. Maximum amplifications of the non-rolling trim occur when the rolling velocity is equal to the nutation rate. This phenomenon is known as "Resonance Instability"[16], and initially one would certainly suspect it as the cause of the catastrophic growth of the pitching and yawing motion. However, these amplification factors can readily be calculated from the linear theory and even approximate values can be obtained when non-linearities in the aerodynamic force and moment system exist.[52] These calculations reveal that an explanation for Catastrophic Yaw cannot be found in current linear or non-linear theory which contains the classical static and dynamic forces and moments from aerodynamics, hydrodynamics, and ballistics. Clearly a new fluid force or forces are required for an understanding and a possible solution.

While dependence on the roll orientation is immediately ruled out in the linear case, its essential role in providing an explanation for roll lock-in suggests that it might profitably be considered in the search for a force and moment which might account for catastrophic yaw. Wind tunnel

tests do reveal two additional effects of roll orientation. First, the normal force and its moment are modified as indicated in Fig. 15 and, second, a side force and moment are found to exist and are indicated in Fig. 16 .

The normal force and its moment may be written*

$$\vec{Z}_w(w) = C_{z_w}\vec{w}\tfrac{1}{2}\rho V^2 S + C_{z_{\gamma,w}}\vec{w}(\sin 2\gamma)\tfrac{1}{2}\rho V^2 S \tag{78}$$

$$\vec{M}_w(w) = C_{m_w}\cdot\vec{w}\tfrac{1}{2}\rho V^2 S d + C_{m_{\gamma,w}}\cdot\vec{w}(\sin 2\gamma)\tfrac{1}{2}\rho V^2 S d \tag{79}$$

where the first term in the right side represents the classical form which in itself may be non-linear in angle of attack and where the second term represents the variation due to roll orientation. Here, also, the coefficient may, in the general case, be a function of angle of attack and roll orientation.

The side force and moment may be written

$$\vec{Z}(\gamma,w) = \pm Z_{\gamma,w}(\sin 4\gamma)\vec{w} = C_{z_{\gamma,w}}\cdot\vec{w}(\sin 4\gamma)\tfrac{1}{2}\rho V^2 S \tag{80}$$

$$\vec{M}(\gamma,w) = M_{\gamma,w}(\sin 4\gamma)\vec{w} = C_{m_{\gamma,w}}\vec{w}(\sin 4\gamma)\tfrac{1}{2}\rho V^2 S d \tag{81}$$

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*The equations (i.e. Eqs. (78 & 79)) are applicable to a cruciform missile only. Similar expressions may be obtained for other rotational symmetries.

In order to determine the contributions of these non-linear forces and moments to the flight performance of missiles and specifically to the dynamic stability of the special pitching and yawing motion, lunar motion, it is necessary to add them to the classical aerodynamics system and to investigate the modified equations of motion. Clearly a general solution of the complete equations of motion containing these new non-linear terms is not possible. However, if we confine our attention to the special case of lunar motion an approximate solution is possible.

Taking the perturbation approach employed in the previous section, the total force and moment, Eqs. (1 - 2), may be extended as

$$\vec{Z} = \left[(\hat{Z}_W + \hat{Z}_{\gamma W}) W + (\hat{Z}_w + \hat{Z}_{\gamma w}) w \right] \frac{\vec{w}}{|\vec{w}|} + Z_{\vec{\zeta}} \vec{\zeta} + Z_{\dot{\vec{w}}} \dot{\vec{w}} + Z_{\dot{\vec{\zeta}}} \dot{\vec{\zeta}} \quad (82)$$

$$\vec{M} = \left[(\hat{M}_W + \hat{M}_{\gamma W}) W + (\hat{M}_w + \hat{M}_{\gamma w}) w \right] \frac{\vec{w}}{|\vec{w}|} + M_{\vec{\zeta}} \vec{\zeta} + M_{\dot{\vec{w}}} \dot{\vec{w}} + M_{\dot{\vec{\zeta}}} \dot{\vec{\zeta}} \quad (83)$$

where

$$\hat{Z}_{\gamma W} = Z_{\gamma_2 W} (\sin 2\gamma) + i Z_{\gamma_4 W} (\sin 4\gamma) \quad (84)$$

$$\hat{Z}_{\gamma w} = Z_{\gamma_2 w} (\sin 2\gamma) + i Z_{\gamma_4 w} (\sin 4\gamma) \quad (85)$$

$$\hat{M}_{\gamma W} = M_{\gamma_4 W} (\sin 4\gamma) - i M_{\gamma_2 W} (\sin 2\gamma) \quad (86)$$

$$\hat{M}_{\gamma w} = M_{\gamma_4 w} (\sin 4\gamma) - i M_{\gamma_2 w} (\sin 2\gamma) \quad (87)$$

In the general case ω and γ vary with time and the coefficients may vary with ω and γ. However, in order to obtain an approximate solution of engineering value it is now assumed that the coefficients are constant and that γ is approximately constant. Accordingly, $\hat{Z}_{\gamma w}$, $\hat{Z}_{\gamma w}$, $\hat{M}_{\gamma w}$, and $\hat{M}_{\gamma w}$ are therefore constant and substitution of Eqs. (82·83) into Eqs. (5·6) yields the differential equations of motion.

The basic form of Eqs. (5-6) after substituting Eqs. (82 - 83) remains unchanged and thus the method of approach, the solution, the analysis, and the discussion of the previous section all apply equally well here.* The new dynamic stability factor, therefore, becomes

$$\lambda^{*} = \frac{Z_{w} + Z_{\gamma w}(\sin 2\gamma)}{2m}(1 \mp \tau) + \frac{M_{q} + u M_{\dot{w}}}{2I}(1 \pm \tau) \pm \frac{u\tau}{I_{x}}\left[M_{pv} + \frac{M_{\gamma w}(\sin 4\gamma)}{p}\right] \qquad \text{([illegible])}$$

The requirement for dynamic stability is obtained from Eq. (80) and is

$$\lambda^{*} < 0 \qquad \text{([illegible])}$$

The contributions of the roll dependent forces and moments to the dynamic stability of Lunar motion may now be considered using Eq. (88). For missiles with canted fins roll lock-in was shown to occur in the region

*Since γ was assumed constant it follows that $p = \omega_{1}$ throughout this development.

$45° \leq \gamma \leq 67\ 1/2°$. Thus the first term in Eq. (88) is reduced in size by the roll dependent force. The effect is noted to be destabilizing in a fin-stabilized missile. The last term in Eq. (88) is the major contributor to stability, and it is noted that the addition of side moment may indeed have a catastrophic effect on the dynamic stability of Lunar motion.

Discussion of Catastrophic Yaw Evaluation

The mathematical analyses of roll lock-in and catastrophic yaw suggest a method for possibly evaluating this type of flight instability and avoiding it in missile design. The method considers the missile to be in pure circular pitching and yawing motion of amplitude W. The first step is to determine if roll lock-in is possible and, if so, to determine the roll lock-in angle. By fluid calculations, where possible, or by experimental model testing in wind tunnels, water tunnels, aeroballistic ranges, etc., values for $C_{l_{\delta_a}}$, C_{l_p} and $C_{l_{\gamma\delta}}$ may be obtained. These values when substituted into Eq. (77) yield values for the roll lock-in angle as a function of the complex angle of attack.

The next step is to compute λ_μ^* from Eq. (88) by using theoretical or experimental values for the required forces and moments and by using the previously determined values for $\gamma(W)$.

Thus a plot of λ_μ^* as a function of the size of the assumed circular motion, W, may be made in which the regions of possible dynamic instability will be revealed. Values of $\lambda_\mu^* > 0$ may be avoided (1) by reducing the initial launching and flight conditions such that unstable

values of W will not be encountered or (2) by configurational or inertia redesign of the missile to assure $\dot{\lambda}_{M} < 0$ for all anticipated values of W. It is clear that either the elimination of roll lock-in by reducing $C_{l_{\gamma\alpha}}$ or the elimination of the detrimental side moment contributions to $\dot{\lambda}_{M}$ may prevent Catastrophic Yaw. It is noted that roll lock-in is a necessary but not a sufficient condition for this instability.

While it is recognized that a missile in flight will not generally have a pure circular motion, as assumed here, one would certainly have considerable lack of confidence in any missile design which was indicated dynamically unstable in this simple mode. The requirement that the missile be dynamically stable in this simple pure circular motion seems mandatory if designs of marginal stability are to be eliminated. This may be particularly true when it is recalled that all flight dynamic instabilities observed by the writer have been observed to be ultimately of the pure circular type.

The extension of this approach to more complicated motions is left for future study. Applications, thus far, of this Catastrophic Yaw Theory are promising and indicate that this simple, easy, and quick method should be of assistance to the design engineer and the flight dynamicist.

Applications of Catastrophic Yaw Theory

Limited theoretical and experimental applications of the Catastrophic Yaw Theory have been made. In the experimental case, an air dropped missile was considered which was occasionally observed to have extremely large pitching and yawing motions accompanied by a failure to roll at the designed rate. The motion was diagnosed to be Catastrophic Yaw and, guided by the

theory given here wind tunnel tests were carried out. Values for the required aerodynamic coefficients were obtained for the basic configuration as well as three modifications. Dynamic stability plots (i.e. $\lambda^{*}_{N}(W)$) were made for each of the four configurations in order to evaluate their relative dynamic stability. Ten full scale designs of each type were constructed and flight tested. The detailed dynamic performance of each type was observed. The results from the observations of flight stability on each type were in the same order of evaluation given by the predictions of the theory. The flight test program is reported in reference (41).

NORC Study

In the analytical case, the exact non-linear differential equations of motion containing induced roll moment and side moment were coded for numerical integration on NORC. The purpose of the NORC calculation was twofold; first, it was desired to compute the flight performance of a particular missile when subjected to various initial launching conditions, and second, it was desired to compare the predictions of the approximate Catastrophic Yaw Theory with the results from the integration of the exact equations.

Approximately 200 complete six degrees of freedom rigid body computations were carried out on NORC. All the in-puts (coefficients, non-linearities, fin incidence angles, physical parameters, initial conditions, etc.) were systemically varied.[53] Three cases most clearly reflect the essential contributions of the induced roll moment and the side moment

to missile flight performance and stability. These three cases are illustrated in Fig. 17 . In order to provide a maximum opportunity for Catastrophic Yaw to occur during the computations, initial conditions were selected which represented Lunar motion. In order to obtain a complete history of the missile's performance for any given case, computations were run both forward and backward in time. As a result, for these three cases the angle of attack and the rolling velocity are all identical at a specific value of time not equal to 0. The "linear case" represents the angle of attack history and the rolling velocity history when only the classical linear aerodynamic system is employed. Here it is noted that the roll performance is regular exhibiting no tendency to lock-in at the Nutation rate, and also exhibiting a pitching and yawing motion that is well damped representing the dynamically stable design. When the induced roll moment is added to the linear calculation, the motions labeled "induced roll moment" are obtained. The pitching and yawing motion is observed to be essentially unchanged as would be predicted from the Catastrophic Yaw Theory. The rolling motion, however, exhibits roll lock-in and later roll break-out. Both of these characteristics are predicted by the Catastrophic Yaw Theory. The roll lock-in occurs when γ is a constant. Roll break-out occurs at that value of the angle of attack for which the induced roll moment is no longer larger than, or equal to, the roll moment due to cant of the fins. Under these circumstances the roll moment due to cant is able to be effective in causing the missile to increase its rolling velocity and tend toward the required steady state value. This case appears to bear out the predictions

of the contribution of the addition of the induced roll moment as affecting the roll performance of missiles. The last case represents the angle of attack and rolling histories when both the induced roll moment and the side moment are added to the linear formulation. Here we observe Catastrophic Yaw and complete roll lock-in. (Fig. 18 is included to illustrate the interesting manner in which the roll orientation angle, γ, changes to time prior to lock-in.)

The three cases clearly indicate the dependence of these quantities in contributing to the overall performance of the missile. Without the induced roll moment, roll lock-in could not occur. Without roll lock-in the side moment could not act constantly in the Magnus sense and cause undamping of the pitching and yawing motion. In order to avoid flight instabilities of this type, it is clear that two obvious avenues of approach are available. One approach is to reduce the side moment to a size such that it will not produce the dynamic instability. The other approach is to reduce the induced roll moment so that roll lock-in does not occur.

A third but not so obvious approach would be to accept the action of the induced roll moment producing roll lock-in, and also accept the side moment, but obtain a value for roll trim in the region $22\ 1/2^{\circ} \leq \gamma_{T} \leq 45^{\circ}$ so that the side moment would then act as a strong stabilizing influence. A value of γ_{T} in this region may be obtained by simply removing the angle of cant from the fins. It is seen from the Catastrophic Yaw Theory that roll lock-in would then occur in this region. Calculations carried out on NORC with and without the roll cant angle are given in Fig. 19 where

it is noted that although roll lock-in occurs when the cant angle is removed the dynamic instability due to side moment does not occur in the pitching and yawing motion.

Concluding Remarks

The results from both the experimental program and the computational study using the exact equations lend considerable confidence in the qualitative predictions of missile dynamic stability using the Catastrophic Yaw Theory.

In addition to providing the aeroballistic design engineer with a quick and easy method for predicting the bounds of missile dynamic stability, the analysis has stimulated the development of wind tunnel and aeroballistic range techniques for the experimental determination of the necessary aerodynamic coefficients, and also has provided general non-linear equations of missile motion in all six degrees of freedom which now are coded for ready computation on NORC and are available for any proposed missile design.[54]

The ever increasing cost and complexity of missile weapons systems requires an extensive use of this new capability which allows the precise computation of missile flight performance and thus system evaluation long before any detailed mechanical or electric design or construction is undertaken.

With this approach unpromising weapons systems designs may be eliminated early in the research and development program and concentrated

emphasis may be placed "a priori" on the most promising systems and their unique problems.

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SYMBOLS

$x\ y\ z$	space fixed axes
$X\ Y\ Z$	Aeroballistic Axes
$\vec{V} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$	missile velocity in space
$\vec{\omega} = \begin{bmatrix} p \\ q \\ r \end{bmatrix}$	missile angular velocity in space
I_x	axial moment of inertia
I	transverse moment of inertia
m	missile mass
$\vec{w} = v + i w = \lvert i\vec{v} \rvert e^{i\theta} = (V + w) e^{i\theta}$ $\quad W \gg \underline{w}$	
$\vec{q} = q + i r$	normal angular velocity
$\frac{d}{dt}$	
$\vec{F} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$	total force on missile
$\vec{M} \begin{bmatrix} L \\ M \\ N \end{bmatrix}$	total moment on missile
$\vec{Z} = Y + i Z$	normal force
$\vec{M} = M + i N$	normal moment

$X_u, Y_u, Z_u, \ldots$
$L_u, M_u, N_u, \ldots$ — Stability Derivatives

$Z_{\dot{w}}, Z_{\dot{q}}, \ldots$
$M_{\dot{w}}, M_{\dot{q}}, \ldots$ — normal stability derivatives

$\hat{Z}_w, \hat{Z}_{\dot{w}}, \hat{Z}_{\delta w}, \hat{Z}_{\delta \dot{w}}$
$\hat{M}_w, \hat{M}_{\dot{w}}, \hat{M}_{\delta w}, \hat{M}_{\delta \dot{w}}$ — nonlinear normal stability derivatives

$L(\gamma\alpha), L(p), L(\delta_n)$ — nonlinear roll moments

$$C_x = \frac{X}{\frac{1}{2}\rho V^2 S} \qquad C_y = \frac{Y}{\frac{1}{2}\rho V^2 S} \qquad C_z = \frac{Z}{\frac{1}{2}\rho V^2 S}$$

$$C_l = \frac{L}{\frac{1}{2}\rho V^2 S d} \qquad C_m = \frac{M}{\frac{1}{2}\rho V^2 S d} \qquad C_n = \frac{N}{\frac{1}{2}\rho V^2 S d}$$

$$S = \frac{\pi}{4} d^2$$

d — missile diameter

$\breve{N}_{1,2,3}$	Eqs. (11-13)
$\breve{\breve{N}}_{1,2,3}$	Eqs. (66-68)
$\underline{K}_{1,2,3}$	Eqs. (25, 26)
$\breve{\breve{K}}_{1,2,3}$	Eqs. (71, 72)
$n_{1,2}$	Eqs. (16)
$\underline{\varphi}_{1,2} = \underline{\lambda}_{1,2} + i\,\underline{\omega}_{1,2}$	Eq (27)
$\breve{\breve{\varphi}}_{1,2} = \bar{\bar{\lambda}}_{1,2} + i\,\breve{\breve{\omega}}_{1,2}$	Eq (70)
$\underline{\lambda}^{*}$	Eq (51)
λ^{*}	Eq (88)
γ	Eq (18)
δ_1	Eq. (77, 69)
$\mathfrak{D}^{(1)}_{N,P}$	Eq (52)
$\mathfrak{D}^{(2)}_{N,P}$	Eq (54)

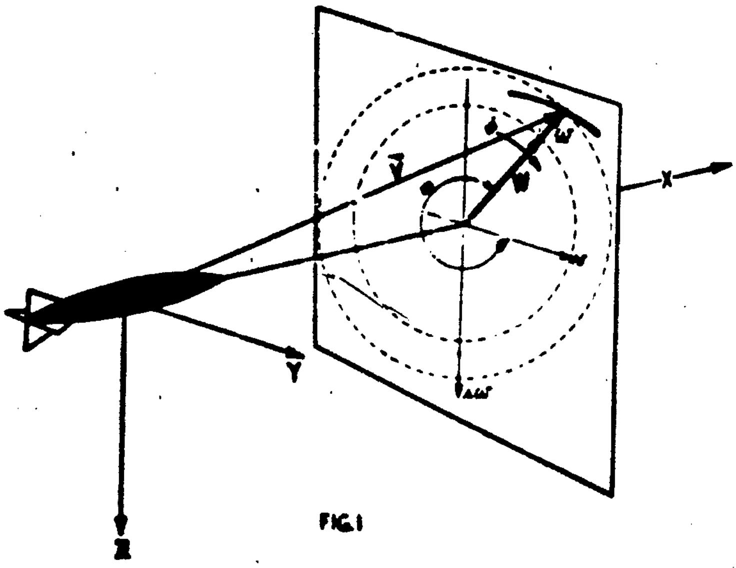


FIG. I

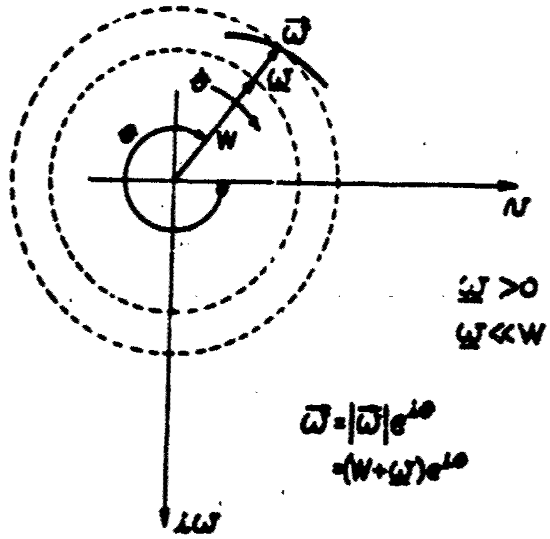


FIG.1a BASIC MOTION

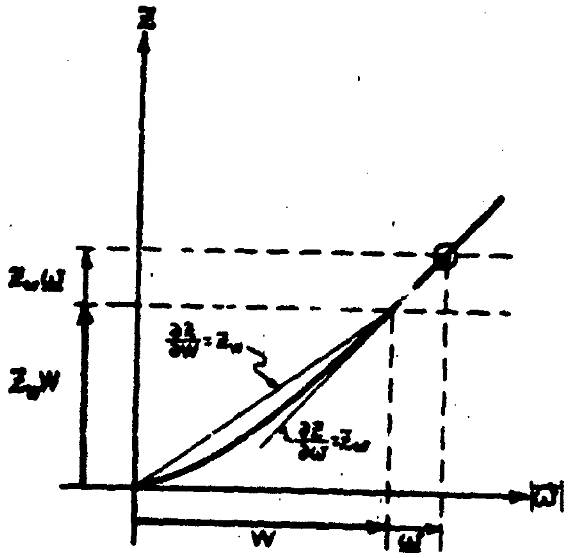


FIG. 2 NONLINEAR STABILITY DERIVATIVES

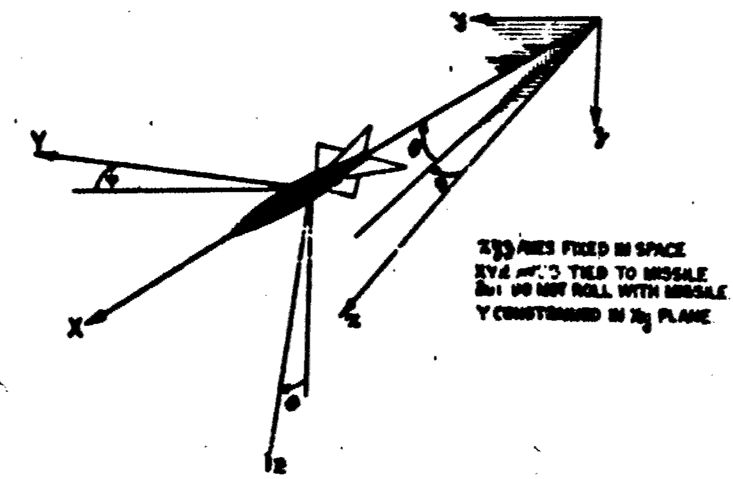


FIG. 3 AEROBALLISTIC AXES

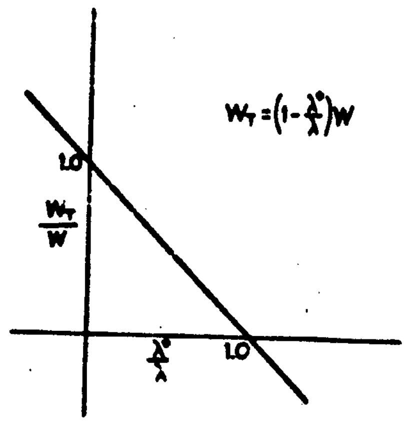
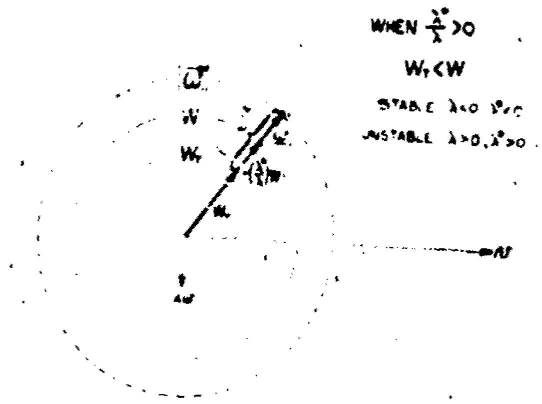


FIG 4 NONLINEAR TRIM



WHEN $\frac{\lambda^*}{\lambda} > 0$
$W_1 < W$
STABLE $\lambda < 0$ $\lambda^* < 0$
UNSTABLE $\lambda > 0, \lambda^* > 0$
W
W_1

FIG 5

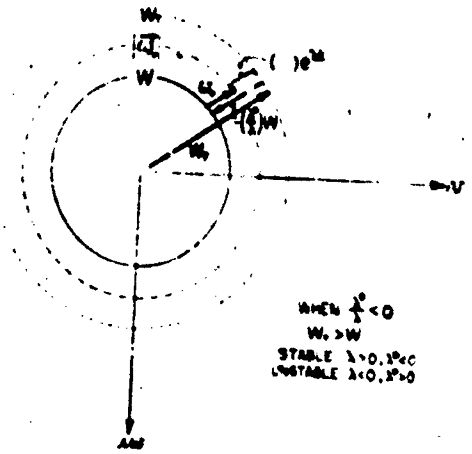

W
V
WHEN $\frac{\lambda^{0}}{\lambda} < 0$
$W_1 > W$
STABLE $\lambda > 0,\ \lambda^{0} < 0$
UNSTABLE $\lambda < 0,\ \lambda^{0} > 0$

FIG. 6

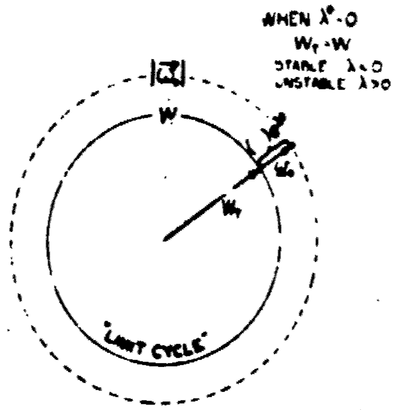


FIG 7

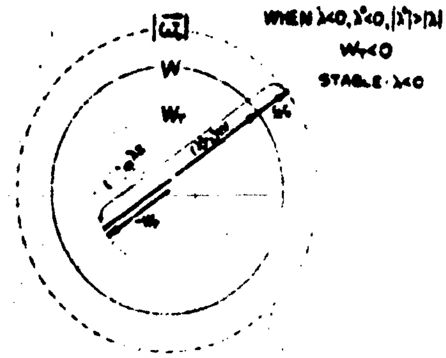
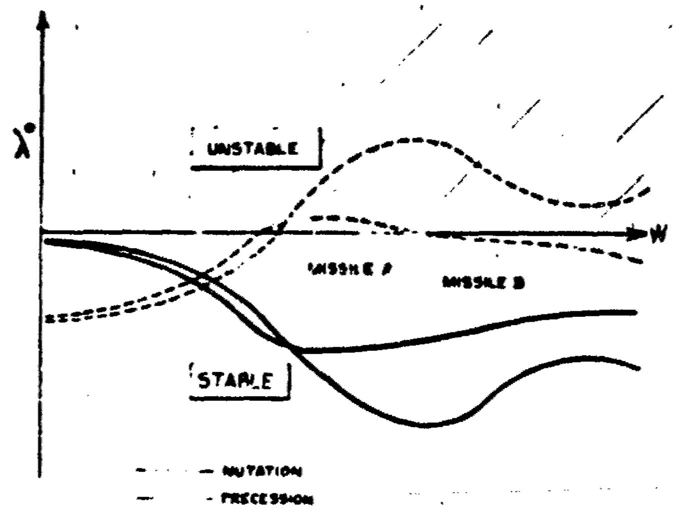
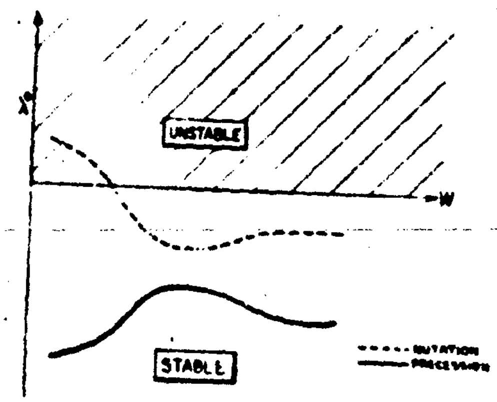


FIG 8

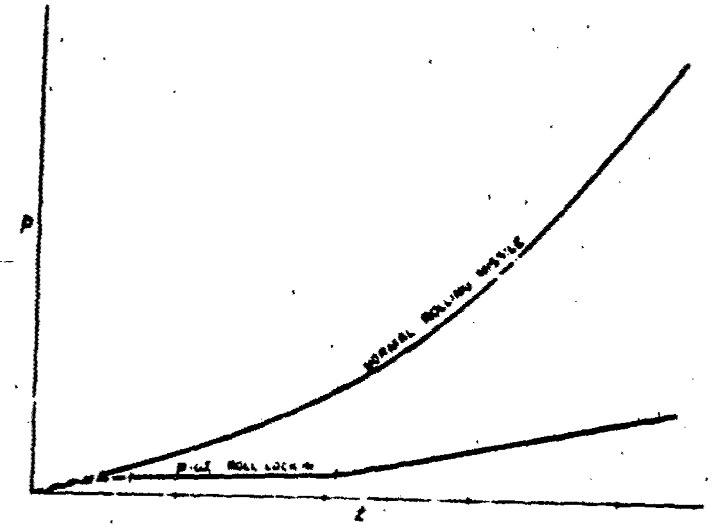


NONLINEAR DYNAMIC STABILITY FOR TWO ROLLING FINNED MISSILES
FIG. 9

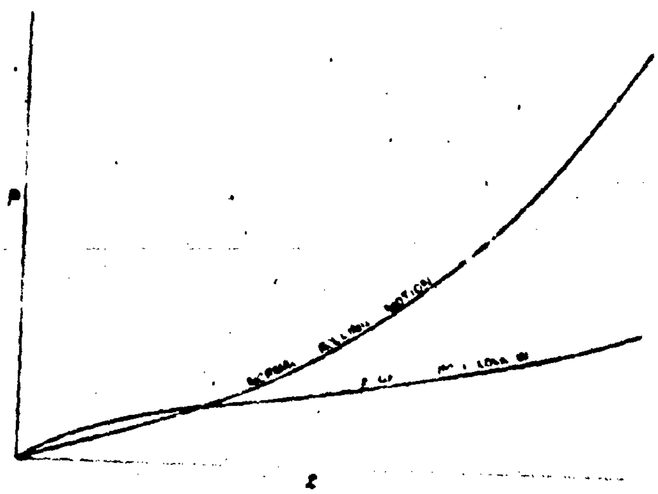


NONLINEAR DYNAMIC STABILITY FOR A SPIN STABILIZED BODY OF REVOLUTION

FIG. 9a



EXPERIMENTALLY OBSERVED ROLL LOCK-IN (MISSILE 1)
FIG 10



EXPERIMENTALLY OBSERVED ROLL LOCK-IN (MISSILE 2)
FIG 10a

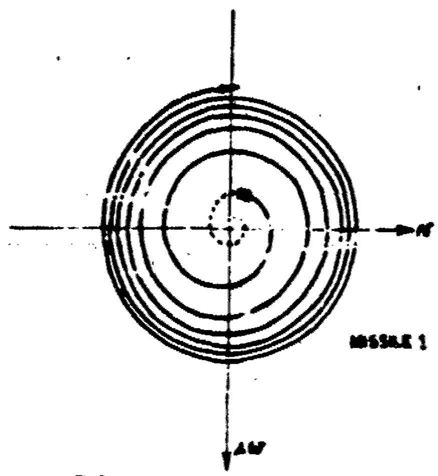


FIG 11 CATASTROPHIC YAW

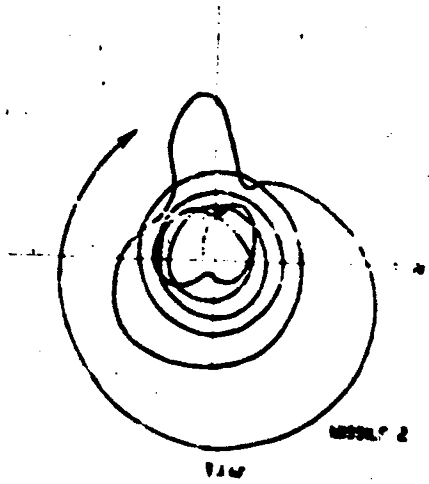
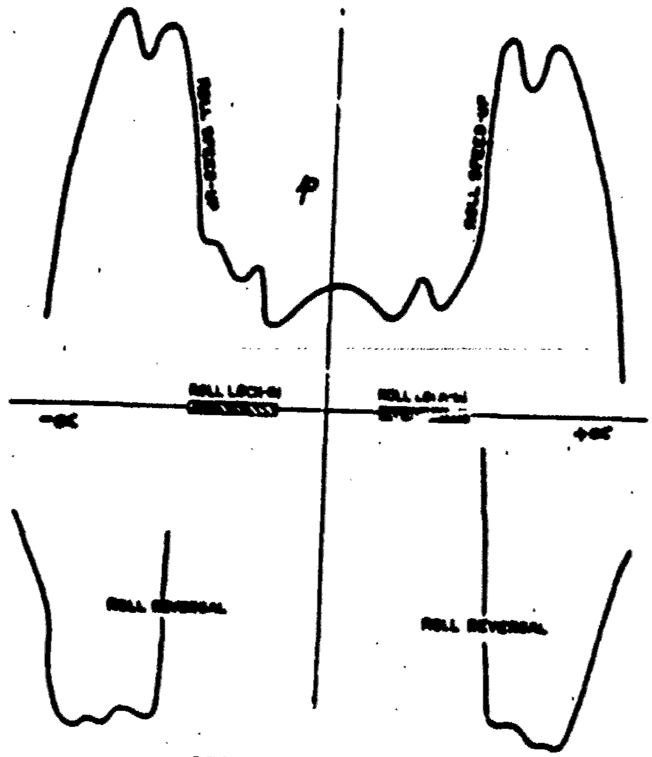
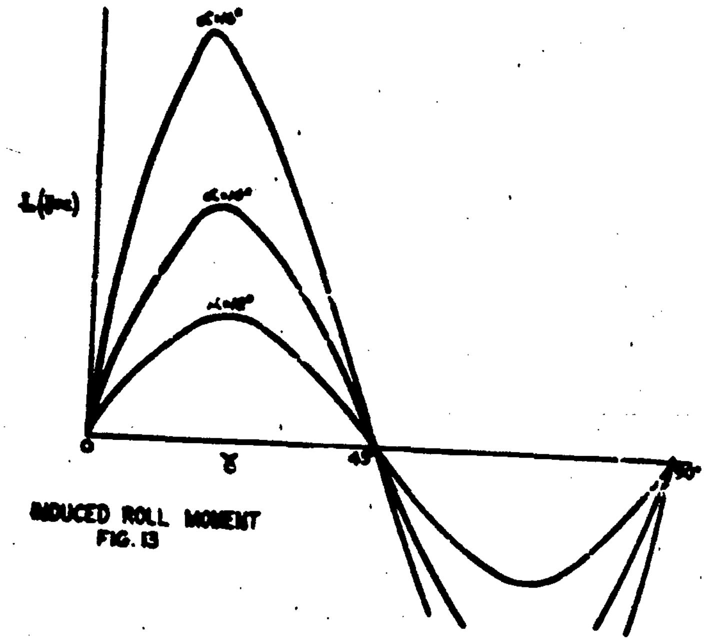


FIG 11. CATASTROPHIC YAW



STEADY STATE ROLLING
FIG. 12



INDUCED ROLL MOMENT
FIG. 13

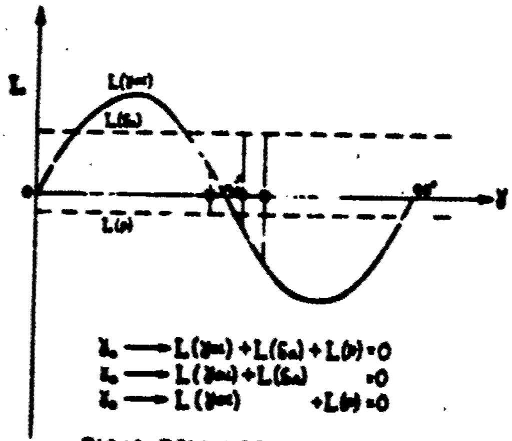
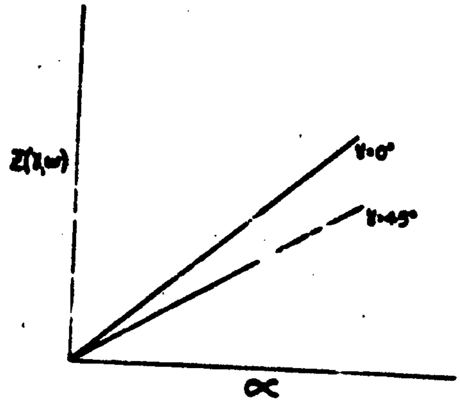


FIG.14 ROLL LOCK-IN



ROLL ORIENTATION EFFECTS ON NORMAL FORCE

FIG. 15

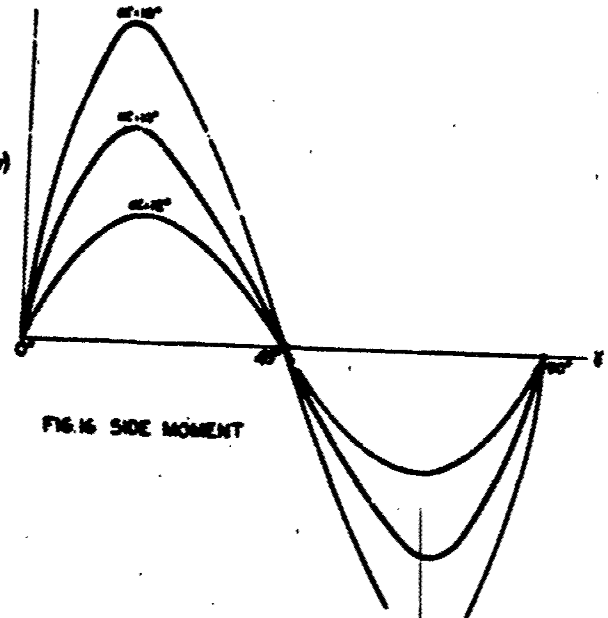
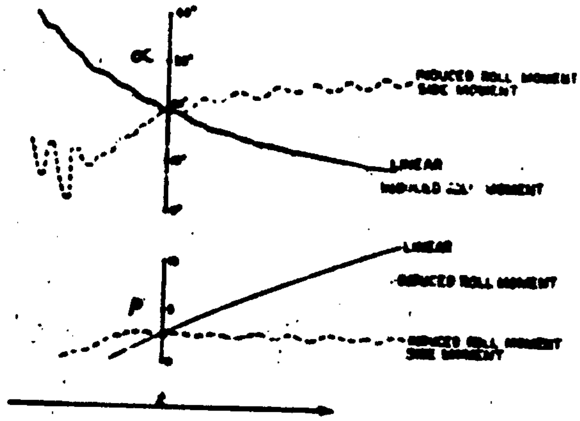


FIG.16 SIDE MOMENT



NORC COMPUTATIONS ON CATASTROPHIC YAW
FIG. 17

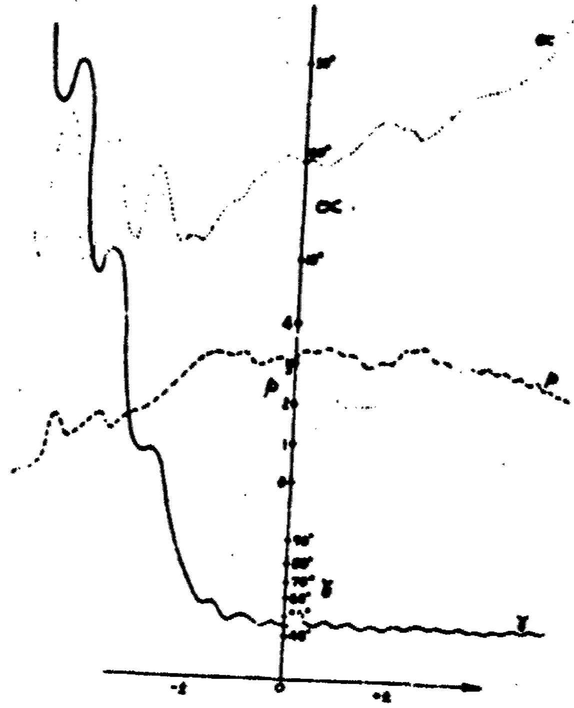


FIG.10 CATASTROPHIC YAW

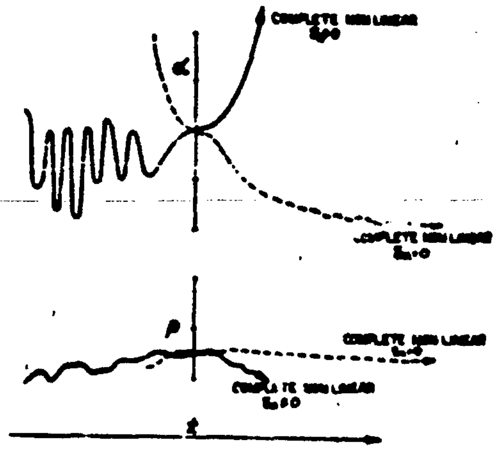


FIG 19 CANT EFFECT ON CATASTROPHIC YAW